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Development and separation of a compressible laminar boundary layer under the action of a very sharp adverse pressure gradiant

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We consider a compressible laminar boundary layer with uniform pressure when $x < x_0$ and a prescribed large adverse pressure gradient when $x > x_0$. The Illingworth–Stewartson transformation is applied, and the transformed external velocity $u_1(x)$ then chosen such that

$$\lambda = -\frac{x_0}{u_0^2} u_1 \frac{du_1}{dx} \frac{T_w}{T_s}$$

is constant, where T_s is the stagnation temperature.

For large λ, when a thin sublayer exists as the layer reacts to the sharp pressure gradient, inner and outer asymptotic expansions are derived and matched for functions F and S which determine the stream function and the temperature. The equations for F and S are largely uncoupled, in that the first approximation to F is independent of S, the first approximation to S depends only on the first approximation to F, and so on.

The skin friction, heat transfer, displacement thickness and momentum thickness are all determined, for $x > x_0$, in terms of $\xi = \lambda(x/x_0 - 1)^{\frac{1}{3}}$, and take the forms

$$\frac{2\nu_0 x}{u_0^3}\left(\frac{\partial u}{\partial y}\right)_w^2 = F_0(\xi) + \sigma_w \lambda^{-1} F_1(\xi) + \sigma_w^2 \lambda^{-2} F_2(\xi) + \ldots,$$

$$-\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}} \frac{(\partial T/\partial y)_w}{T_w - T_s} = G_0(\xi) + \sigma_w \lambda^{-1} G_1(\xi) + \ldots,$$

$$\left(\frac{u_0}{2\nu_0 x}\right)^{\frac{1}{2}} \frac{T_s}{T_w} \delta_1^* = 1{\cdot}2168 + \lambda^{-1} B_1(\xi) - 4{\cdot}2589 \sigma_w \lambda^{-2} \log \lambda \xi^3 - 2{\cdot}4459 \lambda^{-2} \xi^3 + \sigma_w \lambda^{-2} Q(\xi) + 1{\cdot}2168 \sigma_w^2 \lambda^{-2} \xi^3 + \ldots,$$

$$\left(\frac{u_0}{2\nu_0 x_0}\right)^{\frac{1}{2}} \left(\frac{u_1}{u_0}\right)^2 \delta_2^* = 0{\cdot}4696 + 1{\cdot}2168 \lambda^{-2} \xi^3 + \lambda^{-3} C_3(\xi) - 2{\cdot}1295 \sigma_w \lambda^{-4} \log \lambda \xi^6 - 1{\cdot}2230 \lambda^{-4} \xi^6 + \sigma_w \lambda^{-4} C_4(\xi) + 0{\cdot}6084 \sigma_w^2 \lambda^{-4} \xi^6 + \ldots,$$

where $\sigma_w = (T_w - T_s)/T_w$. The various functions $F_0(\xi), F_1(\xi), \ldots, C_4(\xi)$ are all initially given as slowly converging series. By making repeated and extensive use of various properties of flow near a position of boundary-layer separation, the series have all been summed to an accuracy of several significant figures. In particular, it is shown that separation takes place when

$$\xi = 0{\cdot}09766 + 0{\cdot}00403 \sigma_w \lambda^{-1} + 0{\cdot}00035 \sigma_w^2 \lambda^{-2} + \ldots.$$

1. Introduction

This paper considers a compressible boundary layer with a constant pressure p_0 when $x \leqslant x_0$ and a sharp pressure rise when $x > x_0$. The two basic assumptions are made that (i) the Prandtl number σ is equal to unity and (ii) the ratio of the viscosity μ to the absolute temperature T is a function of x alone; thus we have

$$\mu = C(x)\mu_0 T/T_0, \tag{1}$$

where μ_0 and T_0 are values at a suitable reference position. Accordingly we may make a transformation of variables, due to Illingworth (1949) and Stewartson (1949), whereby the equations are partially reduced to incompressible form. *After* transformation, the equations of motion become (Curle & Davies 1971)

$$\partial u/\partial x + \partial v/\partial y = 0, \tag{2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_1\frac{du_1}{dx}(1+S) + \nu_0\frac{\partial^2 u}{\partial y^2}, \tag{3}$$

$$u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} = \nu_0\frac{\partial^2 S}{\partial y^2}, \tag{4}$$

where S is related to the temperature T and is given by

$$S\left(1+\frac{\gamma-1}{2}M_1^2\right) = \frac{T}{T_1} - 1 - \frac{\gamma-1}{2}M_1^2\left(1-\frac{u^2}{u_1^2}\right). \tag{5}$$

With the exception of (1), in which x is measured in the physical plane, x and y otherwise represent distances measured along and normal to the wall in the transformed plane, with associated transformed velocity components u and v. The suffix 1 refers to values at the edge of the boundary layer.

We deduce from (5) that $S \to 0$ at the edge of the boundary layer, where $u \to u_1$ and $T \to T_1$. Likewise, at the wall we have

$$S_w\left(1+\frac{\gamma-1}{2}M_1^2\right) = \frac{T_w}{T_1} - \left(1+\frac{\gamma-1}{2}M_1^2\right),$$

so

$$S_w = T_w/T_s - 1,$$

where

$$T_s = T_1[1+\tfrac{1}{2}(\gamma-1)M_1^2].$$

Thus T_s is the stagnation temperature, which is equal (when $\sigma = 1$) to the wall temperature for which the heat transfer is zero. Thus $S_w = 0$ when there is zero heat transfer, $S_w > 0$ when the wall is heated and $S_w < 0$ when the wall is cooled.

To ensure maximum simplification, the pressure gradient when $x > x_0$ is selected such that

$$\lambda = -\frac{x_0}{u_0^2}u_1\frac{du_1}{dx}(1+S_w) \tag{6}$$

is constant and very large. We thus have a generalization to compressible flow of the problem first studied by Stratford (1954) and later re-examined by the present author (Curle 1976), and the solution reduces to Stratford's when the heat transfer is zero.

When $x \leqslant x_0$, the pressure gradient is zero and u_1 takes the constant value u_0. Then (2)–(4) are readily solved, the velocity components having been given by Blasius (1908) and the temperature function by Pohlhausen (1921). Downstream of $x = x_0$ it is seen from (3) that at the wall (where $u = v = 0$)

$$\nu_0 \left(\frac{\partial^2 u}{\partial y^2}\right)_w = -u_1 \frac{du_1}{dx}(1+S_w) = -\frac{u_0^2}{x_0}\lambda.$$

Thus the velocity profile has a large curvature at the wall, revealing the presence of a thin inner sublayer. Inner and outer asymptotic expansions are therefore obtained and matched both for a stream function ψ and for S, the outer solution being essentially a perturbation of the Blasius–Pohlhausen solution.

The general form of the results may be illustrated by reference to the skin friction, which is determined from the inner expansion and is of the form

$$\tau_w \propto T_0(\xi) + \sigma_w \lambda^{-1} T_1(\xi) + \sigma_w^2 \lambda^{-2} T_2(\xi) + \ldots,$$

where

$$\sigma_w = \frac{S_w}{1+S_w}, \quad \xi = \lambda\left(\frac{x}{x_0} - 1\right)^{\frac{1}{2}}. \tag{7}$$

The functions $T_0(\xi)$, $T_1(\xi)$ and $T_2(\xi)$ are each determined as power series which converge slowly near to separation, owing to the presence of a weak singularity. By using the properties of flow near separation (Goldstein 1948), much as in the incompressible problem (Curle 1976), the separation position and the skin friction have been determined extremely accurately. The separation position is given by

$$\xi = 0{\cdot}0976(6) + 0{\cdot}004030\sigma_w\lambda^{-1} + 0{\cdot}000351\sigma_w^2\lambda^{-2} + \ldots.$$

A similar analysis for the heat transfer reveals that

$$Q_w \propto G_0(\xi) + \sigma_w \lambda^{-1} G_1(\xi) + \ldots.$$

Using the properties of the thermal boundary layer near separation (Buckmaster 1970; Akinrelere 1977), the series for $G_0(\xi)$ and $G_1(\xi)$ are summed. In particular $G_0(\xi)$, which gives the solution for the case of a warm wall, changes from 0·469600 when $\xi = 0$ to 0·216286 at separation. Although $G_0(\xi)$ falls rapidly near to separation, it does not fall to zero, and the value quoted is correct to 5 figures at least.

Analysis of the displacement and momentum thicknesses also leads to slowly convergent series, some of which arose in the incompressible problem. Each further series is summed, using the results of Buckmaster for flow near separation in a compressible boundary layer.

2. The outer solution

As already noted, when $x \leqslant x_0$ the velocity at the edge of the boundary layer takes the constant value u_0, and (2)–(4) are satisfied by the Blasius–Pohlhausen solution. Thus we introduce a stream function ψ, such that $u = \psi_y$ and $v = -\psi_x$, and write

$$\psi = (2u_0\nu_0 x)^{\frac{1}{2}} f_0(\eta), \quad S = S(\eta),$$

where

$$\eta = (u_0/2\nu_0 x)^{\frac{1}{2}} y.$$

It is found (Blasius 1908) that $f_0(\eta)$ satisfies

$$f_0''' + f_0 f_0'' = 0,$$

$$f_0(0) = f_0'(0) = 0, \quad f_0'(\eta) \to 1 \quad \text{as} \quad \eta \to \infty,$$

and (Pohlhausen 1921) that

$$S = S_w(1 - f_0').$$

The boundary-layer approximation will not hold near $x = x_0$, where the pressure gradient is discontinuous, but otherwise, when $x > x_0$, (2)–(4) will apply. The outer solution is a perturbation of the above solution and, following the incompressible analysis (Curle 1976), we write

$$\left.\begin{aligned} \psi &= (2u_0\nu_0 x)^{\frac{1}{2}} F(\xi,\eta) \\ \text{and} \qquad S &= S_w\{1 - F_\eta + S^*(\xi,\eta)\}. \end{aligned}\right\} \tag{8}$$

We substitute into (2)–(4), using the form (6) for the velocity u_1, and find that F and S^* satisfy

$$\xi^2\{F_{\eta\eta\eta} + FF_{\eta\eta}\} = 2\lambda\xi^2(1+\lambda^{-3}\xi^3)\{1+\sigma_w(S^* - F_\eta)\} + \tfrac{2}{3}(\lambda^3+\xi^3)\{F_\eta F_{\eta\xi} - F_\xi F_{\eta\eta}\} \tag{9}$$

and

$$\xi^2(S^*_{\eta\eta} + FS^*_\eta) = 2\lambda\xi^2(1+\lambda^{-3}\xi^3)\{1+\sigma_w(S^* - F_\eta)\} + \tfrac{2}{3}(\lambda^3+\xi^3)\{F_\eta S^*_\xi - F_\xi S^*_\eta\}. \tag{10}$$

Since λ is large, we look for a solution

$$F(\xi,\eta) = f_0(\eta) + \lambda^{-1} f_{10}\xi,\eta) + \lambda^{-2}\log\lambda f_{2L}(\xi\,\eta) + \lambda^{-2} f_2(\xi,\eta) + \lambda^{-3}\log\lambda\,(f_{3L}(\xi,\eta) + \lambda^{-3} f_3(\eta) + \ldots, \tag{11}$$

$$S^*(\xi,\eta) = \lambda^{-1}s_1(\xi,\eta) + \lambda^{-2}\log\lambda\, s_{2L}(\xi,\eta) + \lambda^{-2}s_2(\xi,\eta) + \lambda^{-3}\log\lambda\, s_{3L}(\xi,\eta) + \lambda^{-3}s_3(\xi,\eta) + \ldots. \tag{12}$$

Substituting into (9), and comparing like powers of λ and $\log\lambda$, yields in turn

$$f_0' f_{1\eta\xi} - f_0'' f_{1\xi} = 0, \quad f_0' f_{2L\eta\xi} - f_0'' f_{2L\xi} = 0, \tag{13),(14}$$

$$f_0' f_{2\eta\xi} - f_0'' f_{2\xi} = f_{1\xi} f_{1\eta\eta} - f_{1\eta} f_{1\eta\xi} - 3\xi^2(1 - \sigma_w f_0'), \tag{15}$$

$$f_0' f_{3L\eta\xi} - f_0'' f_{3L\xi} = f_{1\eta} f_{2L\eta\eta} - f_{1\eta} f_{2L\eta\xi} + f_{2L\xi} f_{1\eta\eta} - f_{2L\eta} f_{1\eta\xi}, \tag{16}$$

$$f_0' f_{3\eta\xi} - f_0'' f_{3\xi} = f_{1\xi} f_{2\eta\eta} - f_{1\eta} f_{2\eta\xi} + f_{2\xi} f_{1\eta\eta} - f_{2\eta} f_{1\eta\xi} + 3\sigma_w\xi^2(f_{1\eta} - s_1), \tag{17}$$

whilst (10) yields

$$f_0' s_{1\xi} = 0, \quad f_0' s_{2L\xi} = 0, \tag{18),(19}$$

$$f_0' s_{2\xi} = f_{1\xi} s_{1\eta} - f_{1\eta} s_{1\xi} - 3\xi^2(1 - \sigma_w f_0'), \tag{20}$$

$$f_0' s_{3L\xi} = f_{1\xi} s_{2L\eta} - f_{1\eta} s_{2L\xi} + f_{2L\xi} s_{1\eta} - f_{2L\eta} s_{1\xi}, \tag{21}$$

$$f_0' s_{3\xi} = f_{1\xi} s_{2\eta} - f_{1\eta} s_{2\xi} + f_{2\xi} s_{1\eta} - f_{2\eta} s_{1\xi} + 3\sigma_w\xi^2(f_{1\eta} - s_1). \tag{22}$$

These equations are to some extent uncoupled, since none of (13)–(16) depend upon the s functions, and we may solve the equations successively. Thus

$$f_1 = B(\xi) f_0', \quad f_{2L} = C(\xi) f_0', \tag{23),(24}$$

$$f_2 = \tfrac{1}{2}B^2 f_0'' + D(\xi) f_0' + \xi^3 g_1 + \sigma_w \xi^3 h_1, \tag{25}$$

$$f_{3L} = BC f_0'' + E(\xi) f_0', \tag{26}$$

$$f_3 = \tfrac{1}{6}B^3 f_0''' + BD f_0'' + P(\xi) f_0' + \xi^3 B g_1' + \sigma_w \xi^3 B h_1', \tag{27}$$

and

$$s_1 = 0, \quad s_{2L} = 0, \quad s_2 = \sigma_w \xi^3 - \xi^3/f_0', \tag{28)-(30}$$

$$s_{3L} = 0, \quad s_3 = \xi^3 B f_0''(f_0')^{-2}. \tag{31), (32}$$

Here B, C, D, E and P are arbitrary functions of ξ, and the function $g_1(\eta)$ arose in the incompressible problem (Curle 1976). The function h_1 satisfies

$$f_0' h_1' - f_0'' h_1 = f_0', \tag{33}$$

and it may be shown that

$$h_1 = \eta \log \eta + O(\eta^4 \log \eta) \quad \text{for small } \eta.$$

Numerical integration of (33) shows that

$$h_1 \sim \eta - 0{\cdot}301753 \quad \text{as} \quad \eta \to \infty.$$

It is easily shown that these outer solutions cannot satisfy the boundary conditions at $\eta = 0$ for any choice of the arbitrary functions. As in the incompressible problem, when the boundary layer reacts to the sharp pressure gradient there is a thin inner layer in which η is not the appropriate scale. The correct scale normal to the wall is obtained by writing

$$z = \lambda \xi^{-1} \eta.$$

Before seeking the inner solution, we shall note the outer boundary conditions thereon. Thus we take (11), substitute for f_1, f_{2L}, etc. from (23)–(27), expand for small η and rewrite in terms of the inner co-ordinate z. Likewise we take (12) and substitute for s_1, s_{2L}, etc. from (18)–(22). This yields

$$\begin{aligned} F \sim\ & \lambda^{-2}\{\tfrac{1}{2}\alpha z^2 \xi^2 + \alpha z B \xi + \tfrac{1}{2}\alpha B^2 + \alpha^{-1}\xi^3\} \\ & + \lambda^{-3} \log \lambda \{\xi z + B\}\{\alpha C - \sigma_w \xi^3\} \\ & + \lambda^{-3}\{\sigma_w \xi^4 z(\log \xi + \log z) + \sigma_w \xi^3 B(\log \xi + \log z + 1) + \alpha \xi D z + \alpha B D\} \\ & + \lambda^{-4} \log \lambda \{\alpha \xi E z + \ldots\} + \lambda^{-4}\{\alpha \xi P z + \ldots\} + \ldots \end{aligned} \tag{34}$$

and

$$S^* \sim \lambda^{-1}\{-\alpha^{-1}\xi^2 z^{-1} + \alpha^{-1}\xi B z^{-2} \ldots\} + \lambda^{-2}\{\sigma_w \xi^3 + \ldots\} + \ldots. \tag{35}$$

These boundary conditions indicate the form of the inner solution, which we now investigate.

3. The inner solution

To derive the equations for the inner solution we rewrite (9) and (10) in terms of the new variables ξ and z, and find that

$$\begin{aligned} (1 + \lambda^{-3}\xi^3)^{-1}\{F_{zzz} + \lambda^{-1}\xi F F_{zz}\} = {} & 2\lambda^{-2}\xi^3\{1 + \sigma_w(S^* - \lambda \xi^{-1} F_z)\} \\ & + \tfrac{4}{3}\lambda^2\{\xi^{-1} F_z F_{\xi z} - \xi^{-1} F_{zz} F_\xi - \xi^{-2} F_z^2\} \end{aligned} \tag{36}$$

and

$$\xi(1 + \lambda^{-3}\xi^3)^{-1}\{S^*_{zz} + \lambda^{-1}\xi F S^*_z\} = 2\lambda^{-1}\xi^3\{1 + \sigma_w(S^* - \lambda\xi^{-1}F_z)\} + \tfrac{4}{3}\lambda^2(F_z S^*_\xi - F_\xi S^*_z). \tag{37}$$

Guided by the boundary conditions (34) and (35) we seek a solution

$$\begin{aligned} F &= \lambda^{-2} F_0^*(\xi, z) + \lambda^{-3} F_1^*(\xi, z) + \lambda^{-4} F_2^*(\xi, z) + \ldots, \\ S^* &= \lambda^{-1} S_1^*(\xi, z) + \lambda^{-2} S_2^*(\xi, z) + \ldots. \end{aligned} \tag{38}$$

We have *not* included terms such as $\lambda^{-3}\log\lambda\, F^*_{1L}(\xi, z)$ in this expansion. If such a term is included, the equation for F_{1L} is homogeneous, so there is a solution $F^*_{1L} \equiv 0$. This is consistent with the boundary condition (34) provided that we choose $C(\xi)$ to be

$$C(\xi) = \sigma_w \alpha^{-1}\xi^3. \tag{39}$$

It also follows, if there is no term $\lambda^{-4}\log\lambda\, F^*_{2L}(z)$, that

$$E(\xi) = 0. \tag{40}$$

Upon substituting from (38) into (36) and (37), it is found that F^*_0, F^*, F^*_2, S^*_1 and S^*_2 satisfy the equations

$$F^*_{0zzz} + \tfrac{2}{3}\xi^{-1}\{F^*_{0zz}F^*_{0\xi} - F^*_{0\xi z}F^*_{0z}\} + \tfrac{2}{3}\xi^{-2}F^{*2}_{0z} = 2\xi^3, \tag{41}$$

$$F^*_{1zzz} + \tfrac{2}{3}\xi^{-1}\{F^*_{0z}F^*_{1zz} + F^*_{0zz}F^*_{1\xi} - F^*_{0z}F^*_{1\xi z} - F^*_{0\xi z}F^*_{1z}\} + \tfrac{4}{3}\xi^{-2}F^*_{0z}F^*_{1z} = 2\sigma_w\xi^3(S^*_1 - \xi^{-1}F^*_{0z}, \tag{42}$$

$$F^*_{2zzz} + \tfrac{2}{3}\xi^{-1}\{F^*_{0\xi}F^*_{2zz} + F^*_{0zz}F^*_{2\xi} - F^*_{0z}F^*_{2\xi z} - F^*_{0\xi z}F^*_{2z}\} + \tfrac{4}{3}\xi^{-2}F^*_{0z}F^*_{2z}$$
$$= \tfrac{2}{3}\xi^{-1}(F^*_{1zz}F^*_{1\xi} - F^*_{1z}F^*_{1\xi z}) - \tfrac{2}{3}\xi^{-2}F^{*2}_{1z} + 2\sigma_w\xi^3(S^*_2 - \xi^{-1}F^*_{1z}), \tag{43}$$

$$S^*_{1zz} + \tfrac{2}{3}\xi^{-1}(F^*_{0\xi}S^*_{1z} - F^*_{0z}S^*_{1\xi}) = 2\xi^2, \tag{44}$$

and

$$S^*_{2zz} + \tfrac{2}{3}\xi^{-1}(F^*_{0\xi}S^*_{2z} - F^*_{0z}S^*_{2\xi}) = \tfrac{2}{3}\xi^{-1}(F^*_{1z}S^*_{1\xi} - F^*_{1\xi}S^*_{1z}) + 2\sigma_w\xi^2(S^*_1 - \xi^{-1}F^*_{0z}). \tag{45}$$

The above equations are to be solved in turn for F^*_0, S^*_1, F^*_1, S^*_2 and F^*_2.

The equation (41) for F^*_0 is solved subject to the boundary conditions

$$F^*_0 = F^*_{0z} = 0 \quad \text{when} \quad z = 0$$

and, from (34),

$$F^* \sim \tfrac{1}{2}\alpha\xi^2 z^2 + \alpha B\xi z + \tfrac{1}{2}\alpha B^2 + \alpha^{-1}\xi^3 \quad \text{as} \quad z \to \infty,$$

which is precisely the incompressible problem (Curle 1976). A solution has been given in the form

$$F^*_0(\xi, z) = \xi^2 F_0(z) + \xi^3 F_1(z) + \xi^4 F_2(z) + \xi^5 F_3(z) + \xi^6 F_4(z) + \xi^7 F_5(z) + \dots. \tag{46}$$

It may be noted that

$$B(\xi) = \sum_2^\infty a_n \xi^n$$
$$= -6{\cdot}335486\xi^2 - 19{\cdot}214414\xi^3 - 104{\cdot}20558\xi^4 - 684{\cdot}8897\xi^5 - 4980{\cdot}57\xi^6 \dots, \tag{47}$$

and, for evaluating the skin friction,

$$\left.\begin{aligned} F''_0(0) &= 0{\cdot}469600, & F''_1(0) &= -3{\cdot}137148, & F''_2(0) &= -4{\cdot}906484, \\ F''_3(0) &= -23{\cdot}33114, & F''_4(0) &= -144{\cdot}73528, & F''_5(0) &= -1019{\cdot}4626. \end{aligned}\right\} \tag{48}$$

The reason why the equation for F^*_0 takes the incompressible form is that the thickness of the inner layer tends to zero (on a physical scale) as $\lambda \to \infty$ and so, if the heat transfer remains finite, the variation of temperature across the inner layer will become less as λ increases, and the problem becomes an incompressible one as $\lambda \to \infty$.

We now take the equation (44) for S^*_1, noting that the velocity enters only through the incompressible approximation F^*_0. The equation is thus of the type in which temperature differences are sufficiently small that fluid properties such as density and viscosity may be treated as constant. The equation for S_1^* must be solved subject to the boundary conditions

$$S_1^* = 0 \quad \text{when} \quad z = 0$$

and, from (35),

$$S_1^* \sim -\alpha^{-1}\xi^2 z^{-1} + \alpha^{-1}\xi B z^{-2} \ldots \quad \text{as} \quad z \to \infty. \tag{49}$$

We seek a solution

$$S_1^*(\xi, z) = \xi^2 S_1(z) + \xi^3 S_2(z) + \xi^4 S_3(z) + \xi^5 S_4(z) + \xi^6 S_5(z) + \ldots, \tag{50}$$

whence it is found that

$$S_1'' + \tfrac{2}{3}\alpha z^2 S_1' - \tfrac{4}{3}\alpha z S_1 = 2, \tag{51}$$

$$S_2'' + \tfrac{2}{3}\alpha z^2 S_2' - 2\alpha z S_2 = \tfrac{4}{3}F_1' S_1 - 2F_1 S_1', \tag{52}$$

$$S_3'' + \tfrac{2}{3}\alpha z^2 S_3' - \tfrac{8}{3}\alpha z S_3 = 2(F_1' S_2 - F_1 S_2') + \tfrac{4}{3}(F_2' S_1 - 2F_2 S_1'), \tag{53}$$

$$S_4'' + \tfrac{2}{3}\alpha z^2 S_4' - \tfrac{10}{3}\alpha z S_4 = \tfrac{8}{3}F_1' S_3 - 2F_1 S_3' + 2F_2' S_2 - \tfrac{8}{3}F_2 S_2' + \tfrac{4}{3}F_3' S_1 - \tfrac{10}{3}F_3 S_1' \tag{54}$$

and

$$S_5'' + \tfrac{2}{3}\alpha z^2 S_5' - 4\alpha z S_5 = \tfrac{10}{3}F_1' S_4 - 2F_1 S_4' + \tfrac{8}{3}F_2' S_3 - \tfrac{8}{3}F_2 S_3' + 2F_3' S_2 - \tfrac{10}{3}F_3 S_2' + \tfrac{4}{3}F_4' S_1 - 4F_4 S_1'. \tag{55}$$

The boundary conditions are

$$S_1(0) = S_2(0) = S_3(0) = S_4(0) = S_5(0) \ldots = 0,$$

whilst for large z equation (49) yields

$$S_1 \sim \alpha^{-1} z^{-1}, \quad S_n \sim \alpha^{-1} a_n z^{-2} \quad (n \geqslant 2),$$

with the a_n given by (47). It is easy to work out sufficient terms in these asymptotic forms that the equations may be solved in turn numerically, with the outer boundary conditions satisfied at $z = 10$ or so. This was done, and the first derivatives at the wall, required in calculating heat-transfer rates, are

$$\left.\begin{aligned} &S_1'(0) = -2{\cdot}147263, \quad S_2'(0) = -1{\cdot}980930, \quad S_3'(0) = -7{\cdot}483714, \\ &S_4'(0) = -39{\cdot}18331, \quad S_5'(0) = -238{\cdot}2440. \end{aligned}\right\} \tag{56}$$

We now consider the equation (42) for F_1^*. Since the right-hand side is proportional to σ_w, we anticipate a solution of the form

$$F_1^* = \sigma_w\{\xi^4 G_2(z) + \xi^5 G_3(z) + \xi^6 G_4(z) + \xi^7 G_5(z) + \ldots\}. \tag{57}$$

There are no terms in ξ^2 or ξ^3; the equations for $G_0(z)$ and $G_1(z)$ can be solved explicitly in terms of confluent hypergeometric functions to show that G_0 and G_1 are zero when the appropriate boundary conditions are applied. Similar arguments also show that no terms in $\xi^n \log \xi$ can arise in (57). Since the outer boundary condition

$$F_1^* \sim \sigma_w \xi^4(\log \xi + \log z) z + \sigma_w \xi^3 B(\log \xi + \log z + 1) + \alpha D \xi z + \alpha B D$$

appears to contain terms in $\xi^3 \log \xi$ and $\xi^4 \log \xi$, the function $D(\xi)$ must contain $\xi^n \log \xi$ terms to cancel these. Thus

$$D(\xi) = \sigma_w\{-\alpha^{-1}\xi^3 \log \xi + d_3 \xi^3 + d_4 \xi^4 + d_5 \xi^5 + d_6 \xi^6 + \ldots\}. \tag{58}$$

Upon substituting

$$B(\xi) = a_2 \xi^2 + a_3 \xi^3 + a_4 \xi^4 + \ldots,$$

the outer boundary conditions on the various $G_n(z)$ become

$$\left.\begin{aligned} G_2(z) &\sim z\log z+\alpha d_3 z,\\ G_3(z) &\sim a_2\log z+\alpha d_4 z+\alpha a_2 d_3+a_2,\\ G_4(z) &\sim a_3\log z+\alpha d_5 z+\alpha(a_2d_4+a_3d_3)+a_3,\\ G_5(z) &\sim a_4\log z+\alpha d_6 z+\alpha(a_2d_5+a_3d_4+a_4d_3)+a_4,\end{aligned}\right\} \tag{59}$$

and the boundary conditions at $z=0$ are

$$G_2(0)=G_2'(0)=0,\quad G_3(0)=G_3'(0)=0,\ \text{etc.} \tag{60}$$

Upon substituting from (57) into (42) we find that the equations for the G functions are

$$G_2'''+\tfrac{2}{3}\alpha z^2G_2''-\tfrac{8}{3}\alpha zG_2'+\tfrac{8}{3}\alpha G_2=-2\alpha z, \tag{61}$$

$$G_3'''+\tfrac{2}{3}\alpha z^2G_3''-\tfrac{10}{3}\alpha zG_3'+\tfrac{10}{3}\alpha G_3=2(S_1-F_1')+\tfrac{2}{3}(5F_1'G_2'-4F_1''G_2-3F_1G_2''), \tag{62}$$

$$\begin{aligned}G_4'''+\tfrac{2}{3}\alpha z^2G_4''-4\alpha zG_4'+4\alpha G_4=2(S_2-F_2')&+\tfrac{2}{3}(6F_1'G_3'-5F_1''G_3-3F_1G_3'')\\&+\tfrac{2}{3}(6F_2'G_2'-4F_2''G_2-4F_2G_2'')\end{aligned} \tag{63}$$

and

$$\begin{aligned}G_5'''+\tfrac{2}{3}\alpha z^2G_5''-\tfrac{14}{3}\alpha zG_5'+\tfrac{14}{3}\alpha G_5&=2(S_3-F_3')+\tfrac{2}{3}(7F_1'G_4'-6F_1''G_4-3F_1G_4'')\\&+\tfrac{2}{3}(7F_2'G_3'-5F_2''G_3-4F_2G_3'')+\tfrac{2}{3}(7F_3'G_2'-4F_3''G_2-5F_3G_2'').\end{aligned} \tag{64}$$

These equations were solved numerically as before. In the case of (61) the solution was checked analytically, since there is a particular integral $-\frac{1}{12}\alpha z^4$ and the complementary function is again a confluent hypergeometric function. The second derivatives at $z=0$ are

$$G_2''(0)=1\cdot397128,\quad G_3''(0)=3\cdot946802,\quad G_4''(0)=29\cdot16163,\quad G_5''(0)=244\cdot7864. \tag{65}$$

It was deduced successively, from the asymptotic forms of the solutions for large z, that

$$d_3=1\cdot012848,\quad d_4=23\cdot74748,\quad d_5=167\cdot8087,\quad d_6=1391\cdot763. \tag{66}$$

We turn now to the equation (45) for S_2^*. Since all the terms on the right-hand side are proportional to σ_w, and the outer boundary condition is given from (35) by

$$S_2^*\sim\sigma_w\xi^3\quad\text{as}\quad z\to\infty,$$

we anticipate a solution of the form

$$S_2^*(\xi,z)=\sigma_w\{\xi^3T_2(z)+\xi^4T_3(z)+\xi^5T_4(z)+\xi^6T_5(z)\ldots\}.$$

Upon substitution into (45), with F_0^* given by (46), S_1^* by (50) and F_1^* by (57), we find that

$$T_2''+\tfrac{2}{3}\alpha z^2T_2'-2\alpha zT_2=-2\alpha z, \tag{67}$$

$$T_3''+\tfrac{2}{3}\alpha z^2T_3'-\tfrac{8}{3}\alpha zT_3=2(S_1-F_1')+2(F_1'T_2-F_1T_2')+\tfrac{4}{3}(G_2'S_1-G_2S_1'), \tag{68}$$

$$\begin{aligned}T_4''+\tfrac{2}{3}\alpha z^2T_4'-\tfrac{10}{3}\alpha zT_4=2(S_2-F_2')&+\tfrac{2}{3}\{4F_1'T_3-3F_1T_3'+3F_2'T_2-4F_2T_2'\\&+3G_2'S_2-4G_2S_2'+2G_3'S_1-5G_3S_1'\}\end{aligned} \tag{69}$$

and

$$\begin{aligned}T_5''+\tfrac{2}{3}\alpha z^2T_5'-4\alpha zT_5=2(S_3-F_3')&+\tfrac{2}{3}\{5F_1'T_4-3F_1T_4'+4F_2'T_3-4F_2T_3'+3F_3'T_2-5F_3T_2'\\&+4G_2'S_3-4G_2S_3'+3G_3'S_2-5G_3S_2'+2G_4'S_1-6G_4S_1'\}.\end{aligned} \tag{70}$$

The boundary conditions are

$$T_2(0) = T_3(0) = T_4(0) = T_5(0) = 0$$

and

$$T_2 \to 1, \quad T_3, T_4, T_5 \to 0 \quad \text{as} \quad z \to \infty.$$

Equations (67)–(70) are solved numerically in turn, the solution of (67) being readily checked, since this equation may be solved in terms of confluent hypergeometric functions. The first derivatives at $z = 0$ are

$$T_2'(0) = 0{\cdot}790839, \quad T_3'(0) = 0{\cdot}974128, \quad T_4'(0) = 6{\cdot}10936, \quad T_5'(0) = 44{\cdot}7703. \tag{71}$$

We turn finally to the equation (43) for F_2^*, which has a solution of the form

$$F_2^*(\xi, z) = \sigma_w^2\{\xi^6 H_4(z) + \xi^7 H_5(z) + \ldots\},$$

whence the appropriate substitutions and algebra yield

$$H_4''' + \tfrac{2}{3}\alpha z^2 H_4'' - 4\alpha z H_4' + 4\alpha H_4 = 2(T_2 - G_2') + 2G_2'^2 - \tfrac{8}{3}G_2 G_2'' \tag{72}$$

and

$$H_5''' + \tfrac{2}{3}\alpha z^2 H_5'' - \tfrac{14}{3}\alpha z H_5' + \tfrac{14}{3}\alpha H_5 = 2(T_3 - G_3') + \tfrac{2}{3}\{7F_1' H_4' - 6F_1' H_4 - 3F_1 H_4''\} + \tfrac{2}{3}\{7G_2' G_3' - 5G_2'' G_3 - 4G_2 G_3''\}. \tag{73}$$

The boundary conditions at $z = 0$ are that

$$H_4(0) = H_4'(0) = H_5(0) = H_5'(0) = 0,$$

and those for large z follow from (34). Thus

$$F_2^* \sim \alpha \xi P(\xi) z + \ldots,$$

and so

$$H_4''(z), H_5''(z) \to 0 \quad \text{as} \quad z \to \infty.$$

Numerical solution of (72) and (73) shows that

$$H_4''(0) = -0{\cdot}61283, \quad H_5''(0) = -10{\cdot}9829, \tag{74}$$

and the forms of H_4 and H_5 for large z show that

$$P(\xi) = -\sigma_w^2\{8{\cdot}86289\xi^5 + 92{\cdot}2385\xi^6 + \ldots\}.$$

4. Analysis of skin friction and heat transfer

The skin friction and the heat transfer are both determined by conditions close to the wall, so we derive them from the inner solution. The skin friction in the incompressible plane is

$$\tau_w^* = \mu_0 \left(\frac{\partial u}{\partial y}\right)_w = \mu u_0 \left(\frac{u_0}{2\nu_0 x}\right)^{\frac{1}{2}} \lambda^2 \xi^{-2} (F_{zz})_{z=0}$$

$$= \mu_0 u_0 \left(\frac{u_0}{2\nu_0 x}\right)^{\frac{1}{2}} \xi^{-2}\{F_{0zz}^*(\xi, 0) + \lambda^{-1} F_{1zz}^*(\xi, 0) + \lambda^{-2} F_{2zz}^*(\xi, 0) \ldots\}.$$

Upon substituting for F_0^*, F_1^* and F_2^* we have

$$\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}} \frac{\tau_w^*}{\mu_0 u_0} = \{F_0''(0) + \xi F_1''(0) + \xi^2 F_2''(0) + \xi^3 F_3''(0) + \xi^4 F_3''(0) + \xi^5 F_5''(0) \ldots\}$$
$$+ \sigma_w \lambda^{-1}\{\xi^2 G_2''(0) + \xi^3 G_3''(0) + \xi^3 G_4''(0) + \xi^5 G_5''(0) \ldots\}$$
$$+ \sigma_w^2 \lambda^{-2}\{\xi^4 H_4''(0) + \xi^5 H_5''(0) + \ldots\} + \ldots. \tag{75}$$

We note that the skin friction for any fixed value of ξ depends solely upon $\sigma_w \lambda^{-1}$. With the values of the coefficients substituted from (48), (65) and (74), we therefore have

$$\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}} \frac{\tau_w^*}{\mu_0 u_0} = 0\cdot469600 - 3\cdot137148\xi - (4\cdot906484 - 1\cdot397128\sigma_w \lambda^{-1})\xi^2 - (23\cdot33114 - 3\cdot94680\sigma_w\lambda^{-1})\xi^3 - (144\cdot73528 - 29\cdot16163\sigma_w\lambda^{-1} + 0\cdot61283\sigma_w^2\lambda^{-2})\xi^4 - (1019\cdot4626 - 244\cdot7864\sigma_w\lambda^{-1} + 10\cdot9829\sigma_w^2\lambda^{-2})\xi^5 + \ldots . \quad (76)$$

In the same way, the heat transfer may be derived as

$$-\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}} \frac{(\partial T/\partial y)_w}{T_w - T_s} = (\lambda^2\xi^{-2}F_{zz} - \lambda\xi^{-1}S_z^*)_{z=0}$$
$$= \{F_0''(0) + \xi F_1''(0) + \xi^2 F_2''(0)\ldots\} - \{\xi S_1'(0) + \xi^2 S_2'(0)\ldots\} + \sigma_w\lambda^{-1}\{\xi^2 G_2''(0) + \xi^3 G_3''(0)\ldots\} - \sigma_w\lambda^{-1}\{\xi^2 T_2'(0) + \xi^3 T_3'(0)\ldots\}$$
$$= \{0\cdot469600 - 0\cdot989885\xi - 2\cdot925554\xi \; - 15\cdot847429\xi^3 - 105\cdot55197\xi^4 - 781\cdot2186\xi^5 \ldots\} - \sigma_w\lambda^{-1}\{0\cdot606288\xi^2 + 2\cdot972674\xi^3 + 23\cdot05227\xi^4 + 200\cdot0161\xi^5 + \ldots\} + \ldots \quad (77)$$

upon substituting for the coefficients from (48), (56), (65) and (71).

Following the procedure used in the incompressible case, we estimate the separation position by truncating the series (76) successively after two, three, four, five and six terms, which yields values which *decrease* monotonically.

Likewise, upon squaring the series (76), we have

$$\frac{2\nu_0 x}{\mu_0^2 u_0^3}(\tau_w^*)^2 = 0\cdot220524 - 2\cdot946409\xi + (5\cdot233527 + 1\cdot312182\sigma_w\lambda^{-1})\xi^2 + (8\cdot87212 - 5\cdot05915\sigma_w\lambda^{-1})\xi^3 + (34\cdot52469 - 11\cdot08476\sigma_w\lambda^{-1} + 1\cdot37640\sigma_w^2\lambda^{-2})\xi^4 + (179\cdot5804 - 56\cdot9883\sigma_w\lambda^{-1} + 4\cdot5583\sigma_w^2\lambda^{-2})\xi^5 + \ldots .$$

Truncation here yields values which *increase* monotonically towards the same limit. Given the separation position for the incompressible problem,

$$\xi_s = 0\cdot09766,$$

it proves easy to examine the way in which the estimates of ξ_s *change* with $\sigma_w\lambda^{-1}$. For each of the values of $\sigma_w\lambda^{-1} = -1, -\frac{1}{2}, 0, \frac{1}{2}$ and 1, estimates of the change in ξ_s calculated from the $(\tau_w^*)^2$ series converge quickly to a limiting value which is thus determined very accurately. The values of ξ_s are shown in table 1. It should be noted that, although the values of ξ_s are not correct to the number of figures shown, the *changes* in ξ_s are almost certainly correct to this number of figures. Upon fitting these five points to a quartic polynomial, we find that

$$\xi_s = 0\cdot097660\{1 + 0\cdot041269\epsilon + 0\cdot003594\epsilon^2 + 0\cdot000396\epsilon^3 + 0\cdot000041\epsilon^4\}, \quad (78)$$

where $\epsilon = \sigma_w\lambda^{-1}$, and we shall use this formula in all that follows. In view of the magnitudes of the coefficients we may expect it to be accurate even when $|\epsilon|$ is significantly greater than unity.

$\sigma_w \lambda^{-1}$	ξ_s
-1	0·093946
$-\frac{1}{2}$	0·095728
0	0·097660
$\frac{1}{2}$	0·099768
1	0·102084

TABLE 1. Variation of separation position with $\sigma_w \lambda^{-1}$.

Turning now to the skin friction, we take the series for $(\tau_w^*)^2$ and write

$$\xi/\xi_s = \bar{\xi}, \tag{79}$$

where ξ_s is given by (78). It follows that

$$\frac{2\nu_0 x}{\mu_0^2 u_0^3}(\tau_w^*)^2 = F_0(\bar{\xi}) + \sigma_w \lambda^{-1} F_1(\bar{\xi}) + \sigma_w^2 \lambda^{-2} F_2(\bar{\xi}) + \ldots, \tag{80}$$

where

$$F_0(\bar{\xi}) = 0{\cdot}220524 - 0{\cdot}287746\bar{\xi} + 0{\cdot}049915\bar{\xi}^2 + 0{\cdot}008264\bar{\xi}^3 + 0{\cdot}003140\bar{\xi}^4 + 0{\cdot}001595\bar{\xi}^5 \ldots, \tag{81}$$

$$100F_1(\bar{\xi}) = -1{\cdot}187501\bar{\xi} + 1{\cdot}663477\bar{\xi}^2 - 0{\cdot}368914\bar{\xi}^3 - 0{\cdot}048989\bar{\xi}^4 - 0{\cdot}017707\bar{\xi}^5 \ldots, \tag{82}$$

and

$$1000F_2(\bar{\xi}) = -1{\cdot}03419\bar{\xi} + 1{\cdot}47676\bar{\xi}^2 - 0{\cdot}45208\bar{\xi}^3 + 0{\cdot}03599\bar{\xi}^4 - 0{\cdot}00813\bar{\xi}^5 \ldots. \tag{83}$$

The function $F_0(\bar{\xi})$ is basically the square of the skin friction for the incompressible case (Curle 1976), and need not be further considered. The series (82) for $F_1(\bar{\xi})$ converges well except when $\bar{\xi}$ is close to unity. We may rewrite it as

$$100F_1(\bar{\xi})/(1-\bar{\xi}) = -1{\cdot}187501\bar{\xi} + 0{\cdot}475976\bar{\xi}^2 + 0{\cdot}107062\bar{\xi}^3 + 0{\cdot}058073\bar{\xi}^4 + 0{\cdot}040366\bar{\xi}^5 + \ldots, \tag{84}$$

which is likely to be an improvement near to $\bar{\xi} = 1$, since it satisfies the condition $F_1(1) = 0$. The singularity at separation has been studied by Buckmaster (1970), who concludes that the skin friction near $\bar{\xi} = 1$ should include not only terms like $(1-\bar{\xi})^{\frac{1}{2}}$, which are present in incompressible flow (Goldstein 1948), but also terms like $(1-\bar{\xi})^{\frac{1}{2}}\log(1-\bar{\xi})$ together with smaller multiples of weaker singularities. Numerical support for Buckmaster's conclusions has been given by Davies & Walker (1977). In the present analysis, the most severe singularity in the series (84) is thus expected to be a multiple of $\log(1-\bar{\xi})$. By comparing coefficients, we estimate the multiple as $-0{\cdot}1860$, whence (84) becomes

$$100F_1(\bar{\xi})/(1-\bar{\xi}) = -0{\cdot}1860\log(1-\bar{\xi}) - 1{\cdot}373501\bar{\xi} + 0{\cdot}382976\bar{\xi}^2 + 0{\cdot}045062\bar{\xi}^3 + 0{\cdot}011573\bar{\xi}^4 + 0{\cdot}003166\bar{\xi}^5 \ldots, \tag{85}$$

which may be used to calculate $F_1(\bar{\xi})$ even when $\bar{\xi}$ is close to unity.

We may similarly rewrite (83) as

$$\begin{aligned} 1000F_2(\bar{\xi})/(1-\bar{\xi}) &= -1{\cdot}03419\bar{\xi} + 0{\cdot}44257\bar{\xi}^2 - 0{\cdot}00951\bar{\xi}^3 + 0{\cdot}02648\bar{\xi}^4 + 0{\cdot}01835\bar{\xi}^5 + \ldots \\ &\simeq -0{\cdot}09175\log(1-\bar{\xi}) - 1{\cdot}12594\bar{\xi} + 0{\cdot}39670\bar{\xi}^2 - 0{\cdot}04009\bar{\xi}^3 + 0{\cdot}00354\bar{\xi}^4 + \ldots. \end{aligned} \tag{86}$$

$\bar{\xi}$	$10F_0(\bar{\xi})$	$100F_1(\bar{\xi})$	$1000F_2(\bar{\xi})$	$G_0(\bar{\xi})$	$10G_1(\bar{\xi})$
0	2·205242	0	0	0·469600	0
0·1	1·922573	−0·102489	−0·08910	0·459638	−0·00485
0·2	1·650432	−0·173997	−0·15133	0·449013	−0·01035
0·3	1·389459	−0·216944	−0·18928	0·437589	−0·02089
0·4	1·140430	−0·233936	−0·20551	0·425173	−0·03329
0·5	0·904315	−0·227806	−0·20249	0·411486	−0·05004
0·6	0·682379	−0·201709	−0·18275	0·390701	−0·07321
0·7	0·476369	−0·159288	−0·14886	0·373549	−0·10699
0·8	0·288930	−0·105040	−0·10365	0·352555	−0·16147
0·9	0·124833	−0·045474	−0·05077	0·323723	−0·27183
1·0	0	0	0	0·216286	−2·66604

TABLE 2. Values of $F_0(\bar{\xi})$, $F_1(\bar{\xi})$, $F_2(\bar{\xi})$, $G_0(\bar{\xi})$ and $G_1(\bar{\xi})$.

Values of $F_0(\bar{\xi})$, $F_1(\bar{\xi})$ and $F_2(\bar{\xi})$, derived respectively from (81), (85) and (86), are shown in table 2, and may be used to determine the skin friction at any position.

Both Buckmaster (1970) and Davies & Walker (1977) noted that, for the case of a heated wall, the skin friction appears to have a zero very slightly upstream of the true separation point, and a similar phenomenon may be noted here. From the incompressible analysis (Curle 1976) we know that

$$F_0(\bar{\xi}) \sim 0{\cdot}0676(1-\bar{\xi})+\dots \quad \text{as} \quad \bar{\xi}\to 1,$$

and we may deduce from (85) that

$$F_1(\bar{\xi}) \sim (1-\bar{\xi})\{-0{\cdot}001860\log(1-\bar{\xi})-0{\cdot}009295\}.$$

So, ignoring $\sigma_w^2\lambda^{-2}$ terms, we see that τ_w^{*2} is zero not only when $\bar{\xi}=1$ but also when

$$0{\cdot}0676-\sigma_w\lambda^{-1}\{0{\cdot}001860\log(1-\bar{\xi})+0{\cdot}009295\}=0,$$

and this equation has a root $\bar{\xi}<1$ provided that $\sigma_w<0$. For the problem considered here, the pressure gradient itself depends on the wall temperature, which is why the phenomenon occurs here for cooled, rather than heated walls. The location of this earlier position of zero skin friction is easily calculated to be at

$$1-\bar{\xi}\simeq 2\times 10^{-34} \quad \text{when} \quad \sigma_w\lambda^{-1}=-\tfrac{1}{2},$$

at

$$1-\bar{\xi}\simeq 1\times 10^{-18} \quad \text{when} \quad \sigma_w\lambda^{-1}=-1$$

and at

$$1-\bar{\xi}\simeq 1\times 10^{-10} \quad \text{when} \quad \sigma_w\lambda^{-1}=-2.$$

Clearly, as Davies & Walker also observed for their problem, it is impossible (from a practical viewpoint) to distinguish this zero from that at $\bar{\xi}=1$ for values of $\sigma_w\lambda^{-1}$ for which the present analysis holds.

Turning to the heat transfer, we analyse (77), and first consider the 'warm wall' case, for which

$$-\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}}\frac{(\partial T/\partial y)_w}{T_w-T_s}=G_0$$
$$=0{\cdot}469600-0{\cdot}989885\xi-2{\cdot}295554\xi^2-15{\cdot}847429\xi^3-105{\cdot}55197\xi^4-781{\cdot}2186\xi^5\dots$$
$$=0{\cdot}469600-0{\cdot}096672\bar{\xi}-0{\cdot}027902\bar{\xi}^2-0{\cdot}014761\bar{\xi}^3-0{\cdot}009601\bar{\xi}^4-0{\cdot}006940\bar{\xi}^5\dots . \quad (87)$$

It may be inferred, from the work of Stewartson (1962) and Buckmaster (1970) for example, that when $\bar{\xi}$ approaches unity the series (87) will behave like

$$A+B(1-\bar{\xi})^{\frac{1}{4}}+C(1-\bar{\xi})^{\frac{1}{2}}+\dots. \tag{88}$$

The details for the 'warm wall' case are known (Akinrelere 1977) and for the present problem it is straightforward to show that (88) takes the form

$$\beta_1\{1+0{\cdot}628505(1-\bar{\xi})^{\frac{1}{4}}+0{\cdot}377082(1-\bar{\xi})^{\frac{1}{2}}+0{\cdot}240746(1-\bar{\xi})^{\frac{3}{4}}+\dots\}. \tag{89}$$

We accordingly expand (89) in powers of $\bar{\xi}$ and divide into the series (87), which leads to

$$\begin{aligned} G_0(\bar{\xi})/\{1+0{\cdot}628505(1-\bar{\xi})^{\frac{1}{4}}+0{\cdot}377082(1-\bar{\xi})^{\frac{1}{2}}+0{\cdot}240746(1-\bar{\xi})^{\frac{3}{4}}\} \\ = 0{\cdot}209052+0{\cdot}005937\bar{\xi}+0{\cdot}000940\bar{\xi}^2+0{\cdot}000256\bar{\xi}^3 \\ +0{\cdot}000080\bar{\xi}^4+0{\cdot}000016\bar{\xi}^5+\dots. \end{aligned} \tag{90}$$

Values of $G_0(\bar{\xi})$ may be calculated readily to at least five significant figures, and the results are also shown in table 2. We note that, although $G_0(\bar{\xi})$ falls rapidly as $\bar{\xi}$ approaches 1 (G_0 falls by about 33 % when $\bar{\xi}$ changes from 0·90 to 1), it is certainly non-zero at separation.

More generally, (77) gives the heat transfer and, substituting for $\bar{\xi}$ from (78) and (79), we have

$$-\left(\frac{2\nu_0 x}{u_0}\right)^{\frac{1}{2}}\frac{(\partial T/\partial y)_w}{T_w-T_s} = G_0(\bar{\xi})+\sigma_w\lambda^{-1}G_1(\bar{\xi})+\dots,$$

where

$$100G_1(\bar{\xi}) = -0{\cdot}398957\bar{\xi}-0{\cdot}808547\bar{\xi}^2-0{\cdot}459633\bar{\xi}^3-0{\cdot}368187\bar{\xi}^4-0{\cdot}320887\bar{\xi}^5+\dots,$$

again a slowly convergent series. From the work of Buckmaster, we deduce the presence of terms which are singular like $(1-\bar{\xi})^{\frac{1}{4}}\log(1-\bar{\xi})$ and like $(1-\bar{\xi})^{\frac{1}{4}}$, together with smaller multiples of other mildly singular terms. We may estimate the required multiples of these terms by comparing coefficients, and conclude that the series for $100G_1(\bar{\xi})$ includes multiples of approximately

$$22{\cdot}98(1-\bar{\xi})^{\frac{1}{4}}-8{\cdot}15(1-\bar{\xi})^{\frac{1}{4}}\log(1-\bar{\xi}).$$

Upon extracting these terms we have

$$\begin{aligned} 100G_1(\bar{\xi}) = 22{\cdot}98(1-\bar{\xi})^{\frac{1}{4}}-8{\cdot}15(1-\bar{\xi})^{\frac{1}{4}}\log(1-\bar{\xi})-22{\cdot}98-2{\cdot}803957\bar{\xi} \\ -0{\cdot}691672\bar{\xi}^2-0{\cdot}136768\bar{\xi}^3-0{\cdot}034792\bar{\xi}^4-0{\cdot}009556\bar{\xi}^5+\dots, \end{aligned} \tag{91}$$

which leads to the values of $G_1(\bar{\xi})$ shown in table 2. These are mainly much smaller than the values of $G_0(\bar{\xi})$. Very close to $\bar{\xi}=1$, however, $G_0(\bar{\xi})$ falls very rapidly (as observed earlier) whilst $G_1(\bar{\xi})$ rises even more rapidly in magnitude. Thus, at separation,

$$G_0(1)+\sigma_w\lambda^{-1}G_1(1) \simeq 0{\cdot}216286-0{\cdot}266604\sigma_w\lambda^{-1}+\dots,$$

and the heat transfer varies considerably with wall temperature except when $\sigma_w\lambda^{-1}$ is fairly small.

5. Calculation of displacement thickness and momentum thickness

We shall seek to calculate the quantities

$$\delta_1^* = \int_0^\infty \left(1 - \frac{u}{u_1} + S\right) dy,$$

$$\delta_2^* = \int_0^\infty \left(1 - \frac{u}{u_1}\right) dy,$$

which are related to the true displacement and momentum thicknesses by

$$\frac{a_1\rho_1}{a_0\rho_0}\delta_1 = \left(1 + \frac{\gamma-1}{2}M_1^2\right)\delta_1^* + \frac{\gamma-1}{2}M_1^2\delta_2^*,$$

$$\frac{a_1\rho_1}{a_0\rho_0}\delta_2 = \delta_2^*.$$

Taking δ_1^* first, we substitute for S from (8), and further write

$$\frac{u}{u_1} = \frac{u}{u_0}\left(\frac{u_1}{u_0}\right)^{-1} = \left\{1 + \frac{\lambda^{-2}}{1+S_w}\xi^3 + \ldots\right\} F_\eta,$$

whence

$$\left(\frac{u_0}{2\nu_0 x}\right)^{\frac{1}{2}} \delta_1^* = \int_0^\infty \left\{1 - F_\eta - \frac{\lambda^{-2}}{1+S_w}\xi^3 F_\eta + S_w(1 - F_\eta + S^*) + \ldots\right\} d\eta$$

$$= (1+S_w)\lim_{\eta\to\infty}(\eta - F) + \int_0^\infty \left\{S_w S^* - \frac{\lambda^{-2}}{1+S_w}\xi^3 F_\eta + \ldots\right\} d\eta. \tag{92}$$

The integral in (92) must be split into two parts, representing integration over the inner and outer regions. Thus

$$\int_0^\eta S^*\,d\eta = \int_a^\eta S^*\,d\eta + \int_0^a S^*\,d\eta = \int_a^\eta S^*\,d\eta + \lambda^{-1}\xi \int_0^{a\lambda\xi^{-1}} S^*\,dz. \tag{93}$$

In the first integral we substitute for S^* from (12). Upon using (28)–(30) and neglecting terms smaller than order λ^{-2}, we have

$$S^* = \lambda^{-2}\xi^3\{\sigma_w - 1/f_0'\} = \lambda^{-2}\xi^3\left\{\sigma_w - \frac{d}{d\eta}(h_1/f_0')\right\}.$$

Thus

$$\int_a^\eta S^*\,d\eta = \lambda^{-2}\xi^3\left\{\sigma_w(\eta - a) - \frac{h_1(\eta)}{f_0'(\eta)} + \frac{h_1(a)}{f_0'(a)}\right\}. \tag{94}$$

Likewise, in calculating the second integral in (93) we write

$$\int_0^z S^*\,dz = \lambda^{-1}\int_0^z S_1^* dz = \lambda^{-1}\int_0^z \{\xi^2 S_1(z) + \xi^2 S_2(z) + \xi^4 S_3(z) + \xi^5 S_4(z)\ldots\}\,dz, \tag{95}$$

after substitution from (50). Now it may be deduced from the outer boundary conditions on S_1^* that the asymptotic forms of the integrals in (95) are

$$\int_0^z S_1\,dz \sim -\frac{1}{\alpha}\log z + \beta_1 + \ldots, \quad \int_0^z S_2\,dz \sim \beta_2 + \ldots, \quad \int_0^z S_3\,dz \sim \beta_3 + \ldots, \quad \int_0^z S_4 dz \sim \beta_4 + \ldots, \tag{96}$$

$\bar{\xi}$	$10\,B_1(\bar{\xi})$	$100\,Q_1(\bar{\xi})$	$10^4\,C_3(\bar{\xi})$	$10^5\,C_4(\bar{\xi})$
0	0	0	0	0
0·1	0·00623	0·00293	0·0021	0·0025
0·2	0·02578	0·00553	0·0159	0·0190
0·3	0·06017	0·00007	0·0509	0·0602
0·4	0·11144	−0·02047	0·1149	0·1335
0·5	0·18238	−0·06301	0·2136	0·2430
0·6	0·27705	−0·13490	0·3518	0·3894
0·7	0·40204	−0·24443	0·5333	0·5695
0·8	0·56926	−0·40155	0·7617	0·7749
0·9	0·80807	−0·61900	1·0412	0·9917
1·0	1·35724	−0·91403	1·3776	1·1991

TABLE 3. Values of $B_1(\bar{\xi})$, $Q_2(\bar{\xi})$, $C_3(\bar{\xi})$ and $C_4(\bar{\xi})$.

where β_1, β_2, β_3 and β_4 may be deduced from the numerical solutions as

$$\beta_1 = -0{\cdot}56725, \quad \beta_2 = -9{\cdot}7973, \quad \beta_3 = -42{\cdot}664, \quad \beta_4 = -251{\cdot}417. \tag{97}$$

We now substitute into (93), approximate (94) for small a and (95) for large values of $z = a\lambda\xi^{-1}$, and deduce that

$$\int_0^n S^* d\eta = \lambda^{-2}\xi^3\left\{\beta_1+\beta_2\xi+\beta_3\xi^2+\beta_4\xi^3+\ldots+\sigma_w\eta-\frac{h_1(\eta)}{f_0'(\eta)}-\frac{1}{\alpha}(\log\lambda-\log\xi)\right\}.$$

We substitute into (92), take limiting forms as $\eta\to\infty$, then substitute for $C(\xi)$ from (39), for $D(\xi)$ from (58) and (66), and for β_1, β_2 etc. from (97). This leads to

$$\left(\frac{u_0}{2\nu_0 x}\right)^{\frac{1}{2}}\delta_1^*(1+S_w)^{-1} = 1{\cdot}216783-\lambda^{-1}B(\xi)-2\alpha^{-1}\sigma_w\lambda^{-2}\log\lambda\xi^3$$
$$-2{\cdot}445940\lambda^{-2}\xi^3+\sigma_w\lambda^{-2}Q(\xi)+1{\cdot}216783\sigma_w^2\lambda^{-2}\xi^3+\ldots,$$

where $B(\xi)$ is given by (47) as

$$B(\xi) = -6{\cdot}335486\xi^2-19{\cdot}214414\xi^3-104{\cdot}20558\xi^4-684{\cdot}8897\xi^5-4980{\cdot}57\xi^6\ldots,$$

and

$$Q(\xi) = 2\alpha^{-1}\xi^3\log\xi-4{\cdot}617170\xi^3-33{\cdot}54476\xi^4-210{\cdot}4727\xi^5-1643{\cdot}180\xi^6\ldots.$$

As before, using (78) and (79) to rewrite in terms of $\bar{\xi}$, we have

$$(u_0/2\nu_0x)^{\frac{1}{2}}\,\delta_1^*(1+S_w) = 1{\cdot}217683+\lambda^{-1}B_1(\bar{\xi})-0{\cdot}003967\sigma_w\lambda^{-2}\log\lambda\bar{\xi}^3$$
$$-0{\cdot}002278\lambda^{-2}\bar{\xi}^3+\sigma_w\lambda^{-2}Q_1(\bar{\xi})+0{\cdot}001133\sigma_w^2\lambda^{-2}\bar{\xi}^3+\ldots, \tag{98}$$

where

$$10B_1(\bar{\xi}) = 0{\cdot}60425\bar{\xi}^2+0{\cdot}17897\bar{\xi}^3+0{\cdot}09479\bar{\xi}^4+0{\cdot}06084\bar{\xi}^5+0{\cdot}04424\bar{\xi}^6+\ldots, \tag{99}$$

and

$$100Q_1(\bar{\xi}) = 0{\cdot}49874\bar{\xi}^2+0{\cdot}39669\bar{\xi}^3\log\bar{\xi}-1{\cdot}13129\bar{\xi}^3-0{\cdot}14865\bar{\xi}^4$$
$$-0{\cdot}06143\bar{\xi}^5-0{\cdot}03302\bar{\xi}^6+\ldots. \tag{100}$$

The series (99) for $B_1(\bar{\xi})$ arose in the incompressible problem (Curle 1976) and its sum has been reproduced, for convenience, in table 3. The series for $Q_1(\bar{\xi})$ converges tolerably well and its sum is also shown in table 3.

Turning to the momentum thickness, the momentum-integral equation in the transformed plane (Curle & Davies 1971) takes the form

$$\frac{d}{dx}(u_1^2\delta_2^*) = \nu_0\left(\frac{\partial u}{\partial y}\right)_w - u_1\frac{du_1}{dx}\delta_1^*.$$

We substitute for $(\partial u/\partial y)_w$ from (76), for $u_1 du_1/dx$ from (6) and for δ_1^* from (98), and then use (7), (78) and (79) to change the variable from x *to* ξ. The contributions from $(\partial u/\partial y)_w$ and from δ_1^* are partially self-cancelling since, in the incompressible case at any rate, the main singularities in these terms balance exactly. The final series is therefore fairly convergent and, after integrating, we find

$$\left(\frac{u_0}{2\nu_0 x_0}\right)^{\frac{1}{2}}\left(\frac{u_1}{u_0}\right)^2\delta_2^* = 0{\cdot}469600 + 0{\cdot}001133\lambda^{-2}\xi^3 + \lambda^{-3}C_3(\xi) + 0{\cdot}000140\sigma_w\lambda^{-3}\xi^3 + \sigma_w\lambda^{-4}C_4(\xi) + 10^{-5}\{-0{\cdot}185\sigma_w\lambda^{-4}\log\lambda\xi^6 - 0{\cdot}106\lambda^{-4}\xi^6 + \sigma_w^2\lambda^{-4}(1{\cdot}801\xi^3 + 0{\cdot}053\xi^6)\} + \dots, \quad (101)$$

where

$$10^4 C_3(\xi) = 2{\cdot}1870\xi^3 - 1{\cdot}0701\xi^4 + 0{\cdot}2069\xi^5 + 0{\cdot}0328\xi^6 + 0{\cdot}0116\xi^7 + 0{\cdot}0054\xi^8 + \dots \quad (102)$$

and

$$10^5 C_4(\xi) = 2{\cdot}7077\xi^3 - 1{\cdot}7665\xi^4 + 0{\cdot}7993\xi^5 + 0{\cdot}1847\xi^6\log\xi - 0{\cdot}4941\xi^6 - 0{\cdot}0355\xi^7 - 0{\cdot}0094\xi^8 + \dots. \quad (103)$$

These two series are sufficiently convergent to be readily summed, and the sums are shown in table 3.

Numerical integration of the various ordinary differential equations in this paper was carried out by Miss S. Horsburgh and Mrs M. F. McCall.

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